ASSIGNMENT 1 (part 1)

Mathematics for Machine Learning 811 — 17 January 2022

Unless specified otherwise, the problems in this assignment are intended for pen-and-paper and your submission should include steps. I do encourage you to verify your answers in Python.

What you submit must be your own work, and sources other than the lecture material must be cited. Remember to append a signed plagiarism declaration to your submission.

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1. Using Gaussian elimination, find all solutions of $\boldsymbol{Ax} = \boldsymbol{b}$ with

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

2. Determine the inverses of the following matrices if possible:

$$\textbf{(a)} \quad \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \qquad \textbf{(b)} \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

3. Are the following sets of vectors linearly independent?

$$\textbf{(a)} \quad \boldsymbol{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} 3 \\ -3 \\ 8 \end{bmatrix} \qquad \textbf{(b)} \quad \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

4. **(a)** Write $\boldsymbol{y} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ as a linear combination of $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\boldsymbol{x}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

(b) Do the vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ in part (a) form a generating set of $\mathbb{R}^3$?

5. Consider two subspaces U_1 and U_2, where U_1 is spanned by the columns of $\boldsymbol{A}_1$ and U_2 is spanned by the columns of $\boldsymbol{A}_2$ with

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}$$

(a) Determine the dimensions of U_1 and U_2.

(b) Determine bases of U_1 and U_2.

(c) Determine a basis of $U_1 \cap U_2$, that is, the intersection of U_1 and U_2.

6. Consider the linear mapping $\Phi : \mathbb{R}^3 \to \mathbb{R}^4$ with

$$\Phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}$$

(a) Find the transformation matrix $\boldsymbol{A}_\Phi$.

(b) Determine $\text{rk}(\boldsymbol{A}_\Phi)$.

(c) Compute the kernel and image of Φ. What are $\dim(\ker(\Phi))$ and $\dim(\text{Im}(\Phi))$?

7. Implement the modified Richardson iteration method for solving linear systems, in Python. Read up on how the method's one free parameter can be chosen optimally in terms of the eigenvalues of the coefficient matrix, and demonstrate how your code solves one or two example systems of your own choosing.

8. Consider the Euclidean vector space $\mathbb{R}^5$ with the dot product. A subspace $U \subseteq \mathbb{R}^5$ and $\boldsymbol{x} \in \mathbb{R}^5$ are given by

$$U = \text{span}\left(\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 5 \\ 0 \\ 7 \end{bmatrix}\right), \quad \boldsymbol{x} = \begin{bmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

(a) Determine the orthogonal projection $\pi_U(\boldsymbol{x})$.

(b) Determine the distance between $\boldsymbol{x}$ and $\pi_U(\boldsymbol{x})$.

9. Using the Gram-Schmidt method, turn the basis $B = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ of a 2-dimensional subspace $U \subseteq \mathbb{R}^3$ into an orthonormal basis $C = (\boldsymbol{c}_1, \boldsymbol{c}_2)$ of U, where

$$\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

10. Let $x_1, \ldots, x_n > 0$ be n positive real numbers so that $x_1 + \ldots + x_n = 1$. Use the Cauchy-Schwarz inequality and show that

$$\textbf{(a)} \quad \sum_{i=1}^{n} x_i^2 \geq \frac{1}{n} \qquad \sum_{i=1}^{n} \frac{1}{x_i} \geq n^2$$

Hint: think about the dot product in $\mathbb{R}^n$, and choose specific vectors $\boldsymbol{x}, \boldsymbol{y}$ for the Cauchy-Schwarz inequality.